ON THE VANISHING OF (CO)HOMOLOGY OVER LOCAL RINGS

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Abstract. Considering modules of finite complete intersection dimension over commutative Noetherian local rings, we prove (co)homology vanishing results in which we assume the vanishing of *nonconsecutive* (co)homology groups. In fact, the (co)homology groups assumed to vanish may be arbitrarily far apart from each other.

1. Introduction

The study of the rigidity of Tor for modules over commutative Noetherian local rings began with Auslander's 1961 paper [Aus], which focused on torsion properties of tensor products over unramified regular local rings. In this paper Auslander proved his famous rigidity theorem; for a given pair of finitely generated modules over an unramified regular local ring, if one Tor group vanishes, then all subsequent Tor groups vanish. This result was extended to all regular local rings by Lichtenbaum in [Lic], where the ramified case was proved. Then in [Mur] Murthy generalized the rigidity theorem to arbitrary complete intersections, by showing that if $c+1$ consecutive Tor groups vanish for two finitely generated modules over a complete intersection of codimension c, then all the subsequent Tor groups also vanish. Three decades later Huneke and Wiegand proved in [HuW] that given certain length and dimension restrictions on the two modules involved, the vanishing interval may be reduced by one, i.e. one only has to assume that c consecutive Tor groups vanish.

In [Jo1], rather than considering vanishing intervals of lengths determined by the codimension of the complete intersection, Jorgensen used the notion of the *complexity* of a module. He proved that if X is a finitely generated module of complexity c over a complete intersection R, and

$$\operatorname{Tor}_n^R(X,Y) = \operatorname{Tor}_{n+1}^R(X,Y) = \cdots = \operatorname{Tor}_{n+c}^R(X,Y) = 0$$

for a finitely generated module Y and some integer $n > \dim R - \operatorname{depth} X$, then $\operatorname{Tor}_i^R(X,Y) = 0$ for all $i > \dim R - \operatorname{depth} X$. In addition, by imposing the same length and dimension restrictions on X and Y as Huneke and Wiegand did, he showed that the vanishing interval also in this case can be reduced by one. These results drastically generalize those of Murthy, Huneke and Wiegand, since the complexity of a finitely generated module over a complete intersection never exceeds the codimension of the ring. Moreover, given a complete intersection R of codimension $c > 0$ and an integer $t \in \{0, 1, \dots, c\}$, there exist many finitely generated R-modules having complexity t (see, for example, [AGP, 3.1-3.3] and also [Be2, Corollary 2.3]). In particular, there are a lot of modules of infinite projective dimension having complexity strictly less than the codimension of R.

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A fundamental feature shared by all these vanishing results is the assumption that a certain number of *consecutive* Tor groups vanish, although some results assuming the vanishing of two families of consecutive Tor groups were given in [Jo1] and [Jo2]. The reason behind this is the change of rings spectral sequence

$$\operatorname{Tor}_p^{R/(x)}\left(X, \operatorname{Tor}_q^R(Y, R/(x))\right) \underset{p}{\Rightarrow} \operatorname{Tor}_{p+q}^R(X,Y)$$

associated to $R/(x)$-modules X and Y (see [CaE, XVI,§5]), a sequence which, when x is a non-zerodivisor in R, degenerates into a long exact sequence connecting $\operatorname{Tor}_i^{R/(x)}(X,Y)$ and $\operatorname{Tor}_i^R(X,Y)$. In this situation the vanishing of t consecutive Tor groups over $R/(x)$ implies the vanishing of $t-1$ consecutive Tor groups over R, hence when the ring one studies is a complete intersection one may use induction on the codimension and the already known results for modules over hypersurfaces and regular local rings. However, by using induction on the complexity of a module rather than the codimension of the ring, we give in this paper vanishing results for Tor (respectively, Ext) in which we assume the vanishing of *nonconsecutive* Tor groups (respectively, Ext groups). In fact, the (co)homology groups assumed to vanish may be arbitrarily far apart from each other. This is easy to see in the complexity one case; by a result of Eisenbud (see [Eis]) a finitely generated module of complexity one over a complete intersection is eventually syzygy-periodic of period not more than two, hence if one odd and one even (high enough) Tor (respectively, Ext) vanish, then all higher Tor groups (respectively, Ext groups) vanish.

As the proofs in this paper use induction on the complexity of a module, we need to somehow be able to reduce to a situation in which the module involved has complexity one less than the module we started with. A method for doing this was studied in [Be1], where it was shown that over a commutative Noetherian local ring every module of finite complete intersection dimension has so-called "reducible complexity". Therefore we state the main results for modules having finite complete intersection dimension, a class of modules containing all modules over complete intersections.

2. Preliminaries

Throughout we let $(A, \mathfrak{m}, k)$ be a local (meaning also commutative Noetherian) ring, and we suppose all modules are finitely generated. For an A-module M with minimal free resolution

$$\mathbf{F}_M \colon \cdots \to F_2 \to F_1 \to F_0 \to M \to 0,$$

the rank of F_n, i.e. the integer $\dim_k \operatorname{Ext}_A^n(M,k)$, is the nth *Betti number* of M, and we denote this by $\beta_n(M)$. The *complexity* of M, denoted $\operatorname{cx} M$, is defined as

$$\operatorname{cx} M = \inf\{t \in \mathbb{N} \cup \{0\} \mid \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq an^{t-1} \text{ for } n \gg 0\}.$$

In general, the complexity of a module may be infinite, whereas it is zero if and only if the module has finite projective dimension. The nth *syzygy* of M, denoted $\Omega_A^n(M)$, is the cokernel of the map $F_{n+1} \to F_n$, and it is unique up to isomorphism (note that $\Omega_A^0(M) = M$ and $\operatorname{cx} M = \operatorname{cx} \Omega_A^i(M)$ for every $i \geq 0$). Now let N be an A-module, and consider a homogeneous element $\eta \in \operatorname{Ext}_A^*(M,N)$. By choosing a map $f_\eta \colon \Omega_A^{|\eta|}(M) \to N$ representing η, we obtain a commutative pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_A^{|\eta|}(M) \arrow[r] \arrow[d, "f_\eta"] & F_{|\eta|-1} \arrow[r] \arrow[d] & \Omega_A^{|\eta|-1}(M) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & K_\eta \arrow[r] & \Omega_A^{|\eta|-1}(M) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Note that the module K_η is independent, up to isomorphism, of the map f_η chosen as a representative for η. Following [Be1], we say that M has *reducible complexity* if either the projective dimension of M is finite, or if the complexity of M is positive and finite, and there exists a homogeneous element $\eta \in \operatorname{Ext}_A^*(M,M)$ such that $\operatorname{cx} K_\eta < \operatorname{cx} M$, $\operatorname{depth} K_\eta = \operatorname{depth} M$, and K_η has reducible complexity. The cohomological element η is said to *reduce the complexity* of M.

The module M has finite *complete intersection dimension* if there exist local rings R and Q and a diagram $A \to R \twoheadleftarrow Q$ of local homomorphisms such that $A \to R$ is faithfully flat, $R \twoheadleftarrow Q$ is surjective with kernel generated by a regular sequence (such a diagram $A \to R \twoheadleftarrow Q$ is called a *quasi-deformation* of A), and $\operatorname{pd}_Q(R \otimes_A M)$ is finite. Such modules were first studied in [AGP], and the concept generalizes that of virtual projective dimension defined in [Avr]. As the name suggests, modules having finite complete intersection dimension to a large extent behave homologically like modules over complete intersections. Indeed, over a complete intersection $(S, \mathfrak{n})$ *every* module has finite complete intersection dimension; the completion $\widehat{S}$ of S with respect to the $\mathfrak{n}$-adic topology is the residue ring of a regular local ring Q modulo an ideal generated by a regular sequence, and so $S \to \widehat{S} \twoheadleftarrow Q$ is a quasi deformation.

By [Be1, Proposition 2.2] every module of finite complete intersection dimension has reducible complexity, and given such a module the reducing process decreases the complexity by exactly one. Moreover, the following result shows that by passing to a suitable faithfully flat extension of the ring, we may assume that the cohomological element reducing the complexity of the module is of degree two. Moreover, if η is a cohomological element reducing the complexity of the module, then every positive power of η also reduces the complexity.

Lemma 2.1. *Let M be an A-module of finite complete intersection dimension.*

(i) *There exists a quasi-deformation $A \to R \twoheadleftarrow Q$ such that the R-module $R \otimes_A M$ has reducible complexity by an element $\eta \in \operatorname{Ext}_R^2(R \otimes_A M, R \otimes_A M)$.*

(ii) *If $\eta \in \operatorname{Ext}_A^*(M,M)$ reduces the complexity of M, then so does η^t for every $t \geq 1$.*

Proof. (i) By [AGP, Proposition 7.2(2)] there exists a quasi-deformation $A \to R \twoheadleftarrow Q$ in which the kernel of the map $R \twoheadleftarrow Q$ is generated by a Q-regular element, and such that the Eisenbud operator (with respect to Q) on the minimal free resolution of the R-module $R \otimes_A M$ is eventually surjective. This operator is a chain endomorphism corresponding to an element $\eta \in \operatorname{Ext}_R^2(R \otimes_A M, R \otimes_A M)$, and the proof of [Be1, Proposition 2.2(i)] shows that $\operatorname{cx}_R K_\eta = \operatorname{cx}_R(R \otimes_A M) - 1$ and $\operatorname{depth}_R K_\eta = \operatorname{depth}_R(R \otimes_A M)$. It also follows from the proof of [AGP, Proposition 7.2(2)] that there exists a local ring $\overline{Q}$ and an epimorphism $R \twoheadleftarrow \overline{Q}$ factoring through $R \twoheadleftarrow Q$, whose kernel is generated by a $\overline{Q}$-regular sequence, and such that $\operatorname{pd}_{\overline{Q}}(R \otimes_A M)$ is finite. Since $\operatorname{pd}_{\overline{Q}} R < \infty$ and R is a flat A-module, we see that $\operatorname{pd}_{\overline{Q}} \left(\Omega_R^1(R \otimes_A M) \right)$ is finite, and the exact sequence

$$0 \to R \otimes_A M \to K_\eta \to \Omega_R^1(R \otimes_A M) \to 0$$

shows that the same is true for $\operatorname{pd}_{\overline{Q}} K_\eta$. Therefore K_η has finite complete intersection dimension and consequently has reducible complexity, and this shows that the R-module $R \otimes_A M$ has reducible complexity.

(ii) Suppose η^t reduces the complexity of M for some $t \geq 1$, and let $A \to R \twoheadleftarrow Q$ be a quasi deformation for which $\operatorname{pd}_Q(R \otimes_A M)$ is finite. As $\operatorname{pd}_Q R$ is finite, so is $\operatorname{pd}_Q \left(R \otimes_A \Omega_A^i(M) \right)$ for any $i \geq 0$, hence from the exact sequence

$$0 \to R \otimes_A M \to R \otimes_A K_{\eta^t} \to R \otimes_A \Omega_A^{t|\eta|-1}(M) \to 0$$

we see that $\mathrm{pd}_Q(R \otimes_A K_{\eta^t}) < \infty$. Now from [Be1, Lemma 2.3] there exists an exact sequence

$$0 \to \Omega_A^{|\eta|}(K_{\eta^t}) \to K_{\eta^{t+1}} \oplus F \to K_\eta \to 0$$

in which F is a free A-module, and tensoring this sequence with R we see that $\mathrm{pd}_Q(R \otimes_A K_{\eta^{t+1}})$ is finite. Consequently the A-module $K_{\eta^{t+1}}$ has finite complete intersection dimension, in particular its complexity is reducible. Moreover, the above exact sequence gives

$$\mathrm{cx}\, K_{\eta^{t+1}} \leq \max\{\mathrm{cx}\, K_\eta, \mathrm{cx}\, K_{\eta^t}\} < \mathrm{cx}\, M,$$

and this shows that η^{t+1} reduces the complexity of M. $\square$

We end this section with recalling the following facts regarding flat extensions of local rings. Let $S \to T$ be a faithfully flat local homomorphism, and let X and Y be S-modules. If $\mathbf{F}_X$ is a minimal S-free resolution of X, then the complex $T \otimes_S \mathbf{F}_X$ is a minimal T-free resolution of $T \otimes_S X$, and by [EGA, Proposition (2.5.8)] there exist natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_T(T \otimes_S \mathbf{F}_X, T \otimes_S Y) &\simeq T \otimes_S \mathrm{Hom}_S(\mathbf{F}_X, Y), \\ (T \otimes_S \mathbf{F}_X) \otimes_T (T \otimes_S Y) &\simeq T \otimes_S (\mathbf{F}_X \otimes_S Y). \end{aligned}$$

This gives isomorphisms

$$\begin{aligned} \mathrm{Ext}_T^i(T \otimes_S X, T \otimes_S Y) &\simeq T \otimes_S \mathrm{Ext}_S^i(X, Y), \\ \mathrm{Tor}_i^T(T \otimes_S X, T \otimes_S Y) &\simeq T \otimes_S \mathrm{Tor}_i^S(X, Y), \end{aligned}$$

and as T is faithfully S-flat we then get

$$\begin{aligned} \mathrm{Ext}_T^i(T \otimes_S X, T \otimes_S Y) = 0 &\Leftrightarrow \mathrm{Ext}_S^i(X, Y) = 0, \\ \mathrm{Tor}_i^T(T \otimes_S X, T \otimes_S Y) = 0 &\Leftrightarrow \mathrm{Tor}_i^S(X, Y) = 0. \end{aligned}$$

Moreover, there are equalities

$$\begin{aligned} \mathrm{cx}_S\, X &= \mathrm{cx}_T(T \otimes_S X) \\ \mathrm{depth}_S\, X - \mathrm{depth}_S\, Y &= \mathrm{depth}_T(T \otimes_S X) - \mathrm{depth}_T(T \otimes_S Y), \end{aligned}$$

where the one involving depth follows from [Mat, Theorem 23.3].

3. Vanishing results

Throughout this section we fix two nonzero A-modules M and N, with M of complexity c. The first main result shows that the vanishing of Ext for a certain sequence of numbers forces the vanishing of all the higher Ext groups. The integers for which the cohomology groups are assumed to vanish may be arbitrarily far apart from each other.

Theorem 3.1. *Suppose M has finite complete intersection dimension. If there exist an odd number $q \geq 1$ and a number $n > \mathrm{depth}\, A - \mathrm{depth}\, M$ such that $\mathrm{Ext}_A^i(M, N) = 0$ for $i \in \{n, n+q, \dots, n+cq\}$, then $\mathrm{Ext}_A^i(M, N) = 0$ for all $i > \mathrm{depth}\, A - \mathrm{depth}\, M$.*

Proof. We prove this by induction on c. If $c = 0$, then the projective dimension of M is finite and equal to $\mathrm{depth}\, A - \mathrm{depth}\, M$ by the Auslander-Buchsbaum formula, and so the theorem holds in this case. Now suppose c is positive, and write q as $q = 2t - 1$ where $t \geq 1$. By Lemma 2.1(i) there exists a quasi deformation $A \to R \twoheadleftarrow Q$ such that the R-module $R \otimes_A M$ has reducible complexity by an element $\eta \in \mathrm{Ext}_R^2(R \otimes_A M, R \otimes_A M)$. Moreover, as $R \otimes_A M$ has finite complete intersection dimension we see from Lemma 2.1(ii) that the element η^t also reduces the complexity of $R \otimes_A M$.

The exact sequence

$$(\dagger) \qquad 0 \to R \otimes_A M \to K_{\eta^t} \to \Omega_R^q(R \otimes_A M) \to 0$$

induces a long exact sequence

$$\cdots \to \operatorname{Ext}_R^{i+q}({}_RM, {}_RN) \to \operatorname{Ext}_R^i(K_{\eta^t}, {}_RN) \to \operatorname{Ext}_R^i({}_RM, {}_RN) \to \cdots$$

of cohomology groups (where, for an A-module X, the notation ${}_RX$ is short hand for $R \otimes_A X$), and so the vanishing of the $c+1$ cohomology groups $\operatorname{Ext}_A(M,N)$ implies that $\operatorname{Ext}_R^i(K_{\eta^t}, R \otimes_A N) = 0$ for $i \in \{n, n+q, \dots, n+(c-1)q\}$. As in the proof of Lemma 2.1 the R-module K_{η^t} has finite complete intersection dimension, and the equality $\operatorname{depth}_R K_{\eta^t} = \operatorname{depth}_R(R \otimes_A M)$ holds. Therefore, since $\operatorname{depth}_A A - \operatorname{depth}_A M = \operatorname{depth}_R R - \operatorname{depth}_R(R \otimes_A M)$ and $\operatorname{cx} K_{\eta^t} = \operatorname{cx}(R \otimes_A M) - 1 = c-1$, induction gives $\operatorname{Ext}_R^i(K_{\eta^t}, R \otimes_A N) = 0$ for $i > \operatorname{depth}_A A - \operatorname{depth}_A M$. Consequently

$$\operatorname{Ext}_R^i(R \otimes_A M, R \otimes_A N) \simeq \operatorname{Ext}_R^{i+j(q+1)}(R \otimes_A M, R \otimes_A N)$$

for $i > \operatorname{depth}_A A - \operatorname{depth}_A M$ and $j \geq 0$, and by considering all the pairs $(i,j) \in \{(n,c), (n+q, c-1), \dots, (n+cq, 0)\}$ we see that $\operatorname{Ext}_R^i(R \otimes_A M, R \otimes_A N)$ vanishes for $n+cq \leq i \leq n+cq+c$. Then $\operatorname{Ext}_A^i(M,N)$ also vanishes for these $c+1$ consecutive values, and by [Jo2, Corollary 2.3] and [AGP, Theorem 1.4] we get $\operatorname{Ext}_A^i(M,N) = 0$ for $i > \operatorname{depth}_A A - \operatorname{depth}_A M$. $\square$

The next result is a homology version of Theorem 3.1, and the proof is similar.

Theorem 3.2. *Suppose M has finite complete intersection dimension. If there exist an odd number $q \geq 1$ and a number $n > \operatorname{depth} A - \operatorname{depth} M$ such that $\operatorname{Tor}_i^A(M,N) = 0$ for $i \in \{n, n+q, \dots, n+cq\}$, then $\operatorname{Tor}_i^A(M,N) = 0$ for all $i > \operatorname{depth} A - \operatorname{depth} M$.*

Proof. By induction on c, where the case $c = 0$ follows from the Auslander-Buchsbaum formula. Suppose therefore c is positive, and let the notation be as in the proof of Theorem 3.1. The exact sequence $(\dagger)$ induces a long exact sequence

$$\cdots \to \operatorname{Tor}_i^R({}_RM, {}_RN) \to \operatorname{Tor}_i^R(K_{\eta^t}, {}_RN) \to \operatorname{Tor}_{i+q}^R({}_RM, {}_RN) \to \cdots$$

of homology modules, giving $\operatorname{Tor}_i^R(K_{\eta^t}, {}_RN) = 0$ for $i \in \{n, n+q, \dots, n+(c-1)q\}$. By induction $\operatorname{Tor}_i^R(K_{\eta^t}, {}_RN)$ vanishes for all $i > \operatorname{depth}_A A - \operatorname{depth}_A M$, giving an isomorphism

$$\operatorname{Tor}_i^R(R \otimes_A M, R \otimes_A N) \simeq \operatorname{Tor}_{i+j(q+1)}^R(R \otimes_A M, R \otimes_A N)$$

for $i > \operatorname{depth}_A A - \operatorname{depth}_A M$ and $j \geq 0$. As in the previous proof we see that $\operatorname{Tor}_i^R(R \otimes_A M, R \otimes_A N)$ vanishes for $n+cq \leq i \leq n+cq+c$, and therefore $\operatorname{Tor}_i^A(M,N)$ also vanishes for these $c+1$ consecutive values. From [Jo2, Corollary 2.6] and [AGP, Theorem 1.4] we see that $\operatorname{Tor}_i^A(M,N) = 0$ for $i > \operatorname{depth}_A A - \operatorname{depth}_A M$. $\square$

The number of homology and cohomology groups assumed to vanish in the above two theorems cannot be reduced in general; there exist examples of a complete intersection R and modules X and Y for which $\operatorname{cx} X$ consecutive values of $\operatorname{Tor}^R(X,Y)$ or $\operatorname{Ext}_R(X,Y)$ vanish (starting beyond $\dim R$), while not all the higher homology or cohomology groups vanish (see [Jo1, 4.1] for a homology example). We include such an example, which also shows that the distance between the (co)homology groups assumed to vanish cannot be an even integer.

Example. Let k be a field, let $k[[X,Y]]$ be the ring of formal power series in the indeterminates X and Y, and denote the one dimensional hypersurface $k[[X,Y]]/(XY)$ by A. The complex

$$\cdots \to A \xrightarrow{x} A \xrightarrow{y} A \xrightarrow{x} A \to A/(x) \to 0$$

is exact, and is therefore a minimal free resolution of the module $M = A/(x)$. Denoting by N the module $A/(y)$, we see that $\operatorname{Tor}_i^A(M,N)$ vanishes when i is odd and is nonzero when i is non-negative and even, and that $\operatorname{Ext}_A^i(M,N)$ vanishes when i is even and is nonzero when i is positive and odd.

However, when the ring is a complete intersection and one of the modules for which we are computing (co)homology has finite length, we may reduce the number of (co)homology groups assumed to vanish by one. To prove this, we need the following lemma, which is really just a special case of the two theorems to be proven next. Namely, it treats the case when the (co)homology groups assumed to vanish are all consecutive. Note that the homology part of the lemma follows easily from [Jo1, Theorem 2.6], whereas the complexity one case of the cohomology part, for which we include a proof, is more or less similar to the proof of [HuW, Proposition 2.2].

Lemma 3.3. *Let A be a complete intersection, and suppose N has finite length.*

(i) *If there exists a number $n > \dim A - \operatorname{depth} M$ such that $\operatorname{Ext}_A^i(M,N) = 0$ for $n \leq i \leq n+c-1$, then $\operatorname{Ext}_A^i(M,N) = 0$ for all $i > \dim A - \operatorname{depth} M$.*

(ii) *If there exists a number $n > \dim A - \operatorname{depth} M$ such that $\operatorname{Tor}_i^A(M,N) = 0$ for $n \leq i \leq n+c-1$, then $\operatorname{Tor}_i^A(M,N) = 0$ for all $i > \dim A - \operatorname{depth} M$.*

Proof. (i) As length is preserved under faithfully flat extensions, we may assume the ring A is complete and has infinite residue field (if the latter is not the case, then we use the faithfully flat extension $A \to A[x]_{\mathfrak{m}\,A[x]}$, where x is an indeterminate). We argue by induction on c, where the case $c = 0$ follows from the Auslander-Buchsbaum formula. Suppose therefore $c = 1$, and choose, by [Jo1, Theorem 1.3], a local ring R and a non-zerodivisor $x \in R$ such that $A = R/(x)$ and $\operatorname{pd}_R M < \infty$. As $\dim R = \dim A + 1$ and $\operatorname{depth}_R M = \operatorname{depth}_A M$, the Auslander-Buchsbaum formula gives $\operatorname{pd}_R M = \dim A - \operatorname{depth}_A M + 1$.

The change of rings spectral sequence (see [CaE, XVI,§5])

$$\operatorname{Ext}_A^p\left(M, \operatorname{Ext}_R^q(A,N)\right) \underset{p}{\Rightarrow} \operatorname{Ext}_R^{p+q}(M,N)$$

degenerates into a long exact sequence

$$\begin{array}{ccccccc}
0 \to & \operatorname{Ext}_A^1(M,N) & \to & \operatorname{Ext}_R^1(M,N) & \to & \operatorname{Hom}_A(M,N) & \\
\to & \operatorname{Ext}_A^2(M,N) & \to & \operatorname{Ext}_R^2(M,N) & \to & \operatorname{Ext}_A^1(M,N) & \\
 & \vdots & & \vdots & & \vdots & \\
\to & \operatorname{Ext}_A^i(M,N) & \to & \operatorname{Ext}_R^i(M,N) & \to & \operatorname{Ext}_A^{i-1}(M,N) & \\
 & \vdots & & \vdots & & \vdots &
\end{array}$$

connecting the cohomology groups over A to those over R. Denote $\operatorname{pd}_R M$ by d, and by β_m the mth Betti number of M over R. Localizing the minimal R-free resolution

$$0 \to R^{\beta_d} \to \cdots \to R^{\beta_1} \to R^{\beta_0} \to M \to 0$$

with respect to the multiplicatively closed set $\{1, x, x^2, \dots\} \subseteq R$, keeping in mind that $xM = 0$, we obtain an exact sequence

$$0 \to R_x^{\beta_d} \to \cdots \to R_x^{\beta_1} \to R_x^{\beta_0} \to 0$$

from which we conclude that the equality $\sum_{i=0}^d (-1)^i \beta_i = 0$ holds. As $\beta_m = \ell\left(\operatorname{Ext}_R^m(M,k)\right)$, where $\ell(-)$ denotes the length function, we see that $\sum_{i=0}^d (-1)^i \ell\left(\operatorname{Ext}_R^i(M,k)\right) = 0$, and then by induction on length

$\sum_{i=0}^{d}(-1)^i \ell\left(\operatorname{Ext}_R^i(M,X)\right) = 0$ for every R-module X of finite length. Now let $m > \dim A - \operatorname{depth}_A M$ be an integer. By taking the alternate sum of the lengths in the above long exact sequence, ending in $\operatorname{Ext}_R^{m+1}(M,N) = 0$ and using the isomorphism $\operatorname{Hom}_A(M,N) \simeq \operatorname{Hom}_R(M,N)$, we see that the only terms not canceling are $\ell\left(\operatorname{Ext}_A^m(M,N)\right)$ and $\ell\left(\operatorname{Ext}_A^{m+1}(M,N)\right)$. As this alternate sum is zero we get $\ell\left(\operatorname{Ext}_A^m(M,N)\right) = \ell\left(\operatorname{Ext}_A^{m+1}(M,N)\right)$, and it follows immediately that if $\operatorname{Ext}_A^n(M,N) = 0$ for some $n > \dim A - \operatorname{depth}_A M$, then $\operatorname{Ext}_A^i(M,N)$ vanishes for all $i > \dim A - \operatorname{depth}_A M$. This proves the case $c = 1$.

Now suppose $c > 1$. As A is complete there exists an element $\eta \in \operatorname{Ext}_A^2(M,M)$ reducing the complexity of M. The exact sequence

$$0 \to M \to K_\eta \to \Omega_A^1(M) \to 0$$

induces a long exact sequence

$$\cdots \to \operatorname{Ext}_A^{i+1}(M,N) \to \operatorname{Ext}_A^i(K_\eta,N) \to \operatorname{Ext}_A^i(M,N) \to \cdots$$

in Ext, giving $\operatorname{Ext}_A^i(K_\eta,N) = 0$ for $n \leq i \leq n+c-2$. By induction $\operatorname{Ext}_A^i(K_\eta,N)$ vanishes for all $n > \dim A - \operatorname{depth} M$, and the long exact sequence then implies $\operatorname{Ext}_A^i(M,N) = 0$ for all $n > \dim A - \operatorname{depth} M$.

(ii) The homology part is a special case of [Jo1, Theorem 2.6]. It can also be proved by an argument similar to the above. $\square$

We are now ready to prove the results which sharpen Theorem 3.1 and Theorem 3.2 when N has finite length. Namely, it is enough to assume that c (co)homology groups vanish.

Theorem 3.4. *Let A be a complete intersection, and suppose N has finite length. If there exist an odd number $q \geq 1$ and a number $n > \dim A - \operatorname{depth} M$ such that $\operatorname{Ext}_A^i(M,N) = 0$ for $i \in \{n, n+q, \ldots, n+(c-1)q\}$, then $\operatorname{Ext}_A^i(M,N) = 0$ for all $i > \dim A - \operatorname{depth} M$.*

Proof. If $c = 0$ then the vanishing assumption is vacuous, but as M has finite projective dimension the conclusion follows from the Auslander-Buchsbaum formula. The case $c = 1$ follows from Lemma 3.3(i), so suppose $c > 1$ and that the result holds if we replace M by a module having complexity $c - 1$. As before we may suppose A is complete, hence there exists an element $\eta \in \operatorname{Ext}_A^2(M,M)$ reducing the complexity of M. Write q as $q = 2t - 1$, where $t \geq 1$. The element η^t also reduces the complexity of M, and the corresponding exact sequence

$$0 \to M \to K_{\eta^t} \to \Omega_A^q(M) \to 0$$

induces a long exact sequence

$$\cdots \to \operatorname{Ext}_A^{i+q}(M,N) \to \operatorname{Ext}_A^i(K_{\eta^t},N) \to \operatorname{Ext}_A^i(M,N) \to \cdots$$

in Ext. The assumption on the vanishing $\operatorname{Ext}_A(M,N)$ groups gives $\operatorname{Ext}_A^i(K_{\eta^t},N) = 0$ for $i \in \{n, n+q, \ldots, n+(c-2)q\}$, hence by induction $\operatorname{Ext}_A^i(K_{\eta^t},N)$ vanishes for all $n > \dim A - \operatorname{depth} M$. But then for every $i > \dim A - \operatorname{depth} M$ and $j \geq 0$ there is an isomorphism

$$\operatorname{Ext}_A^i(M,N) \simeq \operatorname{Ext}_A^{i+j(q+1)}(M,N),$$

and by considering the pairs $(i,j) \in \{(n,c), (n+q, c-1), \ldots, (n+(c-1)q, 1)\}$ we see that $\operatorname{Ext}_A^i(M,N)$ vanishes for $n + cq + 1 \leq i \leq n + cq + c$. By Lemma 3.3(i) we are done. $\square$

We omit the proof of the homology version of Theorem 3.4, as it is completely analogous to the proofs of Theorem 3.2 and Theorem 3.4.

Theorem 3.5. *Let A be a complete intersection, and suppose N has finite length. If there exist an odd number $q \geq 1$ and a number $n > \dim A - \operatorname{depth} M$ such that $\operatorname{Tor}_i^A(M,N) = 0$ for $i \in \{n, n+q, \dots, n+(c-1)q\}$, then $\operatorname{Tor}_i^A(M,N) = 0$ for all $i > \dim A - \operatorname{depth} M$.*

An obvious question that arises from Theorem 3.1 and Theorem 3.2 is whether or not the gaps between the (co)homology groups assumed to vanish have to be of the *same* length. Consider therefore the following cohomology vanishing condition:

Condition. *There exist positive odd integers $q_1, \dots, q_c$ such that*

$$\operatorname{Ext}_A^n(M,N) = \operatorname{Ext}_A^{n+q_1}(M,N) = \cdots = \operatorname{Ext}_A^{n+q_1+\cdots+q_c}(M,N) = 0$$

for some $n > \operatorname{depth} A - \operatorname{depth} M$.

Does the conclusion of Theorem 3.1 hold if we replace the vanishing assumption in the theorem with the considerably weaker condition stated above? Similarly we may ask if the conclusion of Theorem 3.2 holds if we replace the vanishing assumption in the theorem with the homology version of the above condition.

In case the answer to the above questions are positive, it is possible that a proof similar to those used in the main results exist, i.e. a proof based on reducing the complexity of M. We end this paper with two results showing that the answer is positive when the complexity of M is not more than 2. A proof is provided only for the cohomology version, the proof of the homology version is analogous.

Theorem 3.6. *Suppose M has finite complete intersection dimension and complexity 2. If there exist odd numbers $p \geq 1$ and $q \geq 1$ such that*

$$\operatorname{Ext}_A^n(M,N) = \operatorname{Ext}_A^{n+p}(M,N) = \operatorname{Ext}_A^{n+p+q}(M,N) = 0$$

for some $n > \operatorname{depth} A - \operatorname{depth} M$, then $\operatorname{Ext}_A^i(M,N) = 0$ for all $i > \operatorname{depth} A - \operatorname{depth} M$.

Proof. We argue by induction on the sum $p+q$ (by assumption this sum is at least 2). If $p+q=2$, then $p=q=1$, and we are done by Theorem 3.1. Suppose therefore $p+q>2$, and write p and q as $p = 2s-1$ and $q = 2t-1$, where $s,t \in \mathbb{N}$.

Choose, by Lemma 2.1(i), a quasi deformation $A \to R \twoheadleftarrow Q$ such that the complexity of the R-module $R \otimes_A M$ is reducible by an element $\eta \in \operatorname{Ext}_R^2(R \otimes_A M, R \otimes_A M)$. For an A-module X, denote by ${}_RX$ the R-module $R \otimes_A X$. By Lemma 2.1(ii) the element η^s also reduces the complexity of ${}_RM$, and from its associated exact sequence

$$0 \to {}_RM \to K_{\eta^s} \to \Omega_R^p({}_RM) \to 0$$

we deduce that $\operatorname{Ext}_R^n(K_{\eta^s}, {}_RN) = 0$. However, the R-module K_{η^s} has complexity 1, and therefore, by [AGP, Theorem 7.3], the modules $\Omega_R^i(K_{\eta^s})$ and $\Omega_R^{i+2}(K_{\eta^s})$ are isomorphic for $i > \operatorname{depth}_R R - \operatorname{depth}_R K_{\eta^s}$. Consequently, as $\operatorname{depth}_R K_{\eta^s} = \operatorname{depth}_R({}_RM)$, we see that

$$\operatorname{Ext}_R^{n+2i}(K_{\eta^s}, {}_RN) = 0$$

for every $i \geq 0$, in particular $\operatorname{Ext}_R^{n+q-1}(K_{\eta^s}, {}_RN) = 0$. Now since the above exact sequence yields an exact sequence

$$\operatorname{Ext}_R^{n+q-1}(K_{\eta^s}, {}_RN) \to \operatorname{Ext}_R^{n+q-1}({}_RM, {}_RN) \to \operatorname{Ext}_R^{n+q}(\Omega_R^p({}_RM), {}_RN),$$

whose end terms both vanish, we see that $\operatorname{Ext}_R^{n+q-1}({}_RM, {}_RN) = 0$.

The element $\eta^t \in \operatorname{Ext}_R^*({}_RM, {}_RM)$ also reduces the complexity of ${}_RM$, and from its associated exact sequence

$$0 \to {}_RM \to K_{\eta^t} \to \Omega_R^q({}_RM) \to 0$$

we see that $\operatorname{Ext}_R^{n+p}(K_{\eta^t}, {}_RN) = 0$. As with the module K_{η^s}, the module K_{η^t} is eventually periodic of period 2, and therefore

$$\operatorname{Ext}_R^{n+1+2i}(K_{\eta^t}, {}_RN) = 0$$

for every $i \geq 0$. In particular $\operatorname{Ext}_R^{n+1}(K_{\eta^t}, {}_RN) = 0$, and since the previous exact sequence yields an exact sequence

$$\operatorname{Ext}_R^n({}_RM, {}_RN) \to \operatorname{Ext}_R^{n+1}(\Omega_R^q({}_RM), {}_RN) \to \operatorname{Ext}_R^{n+1}(K_{\eta^t}, {}_RN),$$

we see that $\operatorname{Ext}_R^{n+q+1}({}_RM, {}_RN) = 0$.

We have just shown that both $\operatorname{Ext}_R^{n+q-1}({}_RM, {}_RN)$ and $\operatorname{Ext}_R^{n+q+1}({}_RM, {}_RN)$ vanish. Now since p and q are not equal, either $p > q \geq 1$ or $q > p \geq 1$. Define integers n', p' and q' by

$$(n', p', q') = \begin{cases} (n+q+1, p-q-1, q) & \text{if } p > q \\ (n, p, q-p-1) & \text{if } p < q, \end{cases}$$

Then p' and q' are both odd, the groups $\operatorname{Ext}_R^i({}_RM, {}_RN)$ vanish for $i \in \{n', n'+p', n'+p'+q'\}$, and $p'+q' < p+q$. By induction $\operatorname{Ext}_R^i({}_RM, {}_RN)$ vanishes for $i > \operatorname{depth} A - \operatorname{depth} M$, and consequently $\operatorname{Ext}_A^i(M, N) = 0$ for $i > \operatorname{depth} A - \operatorname{depth} M$. $\square$

Theorem 3.7. *Suppose M has finite complete intersection dimension and complexity* 2*. If there exist odd numbers $p \geq 1$ and $q \geq 1$ such that*

$$\operatorname{Tor}_n^A(M, N) = \operatorname{Tor}_{n+p}^A(M, N) = \operatorname{Tor}_{n+p+q}^A(M, N) = 0$$

for some $n > \operatorname{depth} A - \operatorname{depth} M$, then $\operatorname{Tor}_i^A(M, N) = 0$ for all $i > \operatorname{depth} A - \operatorname{depth} M$.

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